

# Two-loop renormalization of scalar and pseudoscalar fermion bilinears on the lattice

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## Abstract

We compute the two-loop renormalization functions, in the  $\text{RI}'$  scheme, of local bilinear quark operators  $\bar{\psi}\Gamma\psi$ , where  $\Gamma$  denotes the Scalar and Pseudoscalar Dirac matrices, in the lattice formulation of QCD. We consider both the flavor non-singlet and singlet operators; the latter, in the scalar case, leads directly to the two-loop fermion mass renormalization,  $Z_m$ .

As a prerequisite for the above, we also compute the quark field renormalization,  $Z_\psi$ , up to two loops.

We use the clover action for fermions and the Wilson action for gluons. Our results are given as a polynomial in  $c_{SW}$ , in terms of both the renormalized and bare coupling constant, in the renormalized Feynman gauge. We also confirm the 1-loop renormalization functions, for generic gauge.

Finally, we present our results in the  $\overline{MS}$  scheme, for easier comparison with calculations in the continuum.

The corresponding results, for fermions in an *arbitrary* representation, are included in an Appendix.

**Keywords:** Lattice QCD, Lattice perturbation theory, Fermion bilinears, clover action.

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## I. INTRODUCTION

Studies of hadronic properties using the lattice formulation of QCD rely on the computation of matrix elements and correlation functions of composite operators, made out of quark fields. A whole variety of such operators has been considered and studied in numerical simulations, including local and extended bilinears, and four-fermi operators. A proper renormalization of these operators is most often indispensable for the extraction of physical results from the lattice.

In this work we study the renormalization of fermion bilinears  $\mathcal{O} = \bar{\psi}\Gamma\psi$  on the lattice, where  $\Gamma = \mathbb{1}, \gamma_5$ . We consider both flavor singlet and nonsinglet operators. The cases  $\Gamma = \gamma_\mu, \gamma_5 \gamma_\mu, \gamma_5 \sigma_{\mu\nu}$ , will be presented in a sequel to this work. In order to obtain the renormalization functions of fermion bilinears we also compute the quark field renormalization,  $Z_\psi$ , as a prerequisite.

We employ the standard Wilson action for gluons and clover-improved Wilson fermions. The number of quark flavors  $N_f$ , the number of colors  $N_c$  and the clover coefficient  $c_{\text{SW}}$  are kept as free parameters.

Our two-loop calculations have been performed in the bare and in the renormalized Feynman gauge. For 1-loop quantities, the gauge parameter is allowed to take arbitrary values.

The main results presented in this work are the following 2-loop bare Green's functions (amputated, one-particle irreducible (1PI)):

- Fermion self-energy:  $\Sigma_\psi^L(q, a_L)$
- 2-pt function of the scalar  $\bar{\psi}\psi$  :  $\Sigma_S^L(qa_L)$
- 2-pt function of the pseudoscalar  $\bar{\psi}\gamma_5\psi$  :  $\Sigma_P^L(qa_L)$

( $a_L$  : lattice spacing,  $q$  : external momentum)

In general, one can use bare Green's functions to construct  $Z_{\mathcal{O}}^{X,Y}$ , the renormalization function for operator  $\mathcal{O}$ , computed within a regularization  $X$  and renormalized in a scheme  $Y$ .

We employ two widely used schemes to compute the various 2-loop renormalization functions:

- The  $RI'$  scheme:  $Z_\psi^{L,RI'}, Z_S^{L,RI'}, Z_P^{L,RI'}$
- The  $\overline{MS}$  scheme:  $Z_\psi^{L,\overline{MS}}, Z_S^{L,\overline{MS}}, Z_P^{L,\overline{MS}}$

The flavor singlet scalar renormalization function is equal to the fermion mass multiplicative renormalization,  $Z_m$ , which is an essential ingredient in computing quark masses.

For convenience, the results for  $Z_{\mathcal{O}}^{X,Y}$  are given in terms of both the bare coupling constant  $g_0$  and the renormalized one:  $g_{RI'}, g_{\overline{MS}}$ .

Finally, as one of several checks on our results, we construct the 2-loop renormalized Green's functions in  $RI'$ :  $\Sigma_{\mathcal{O}}^{RI'}(q, \bar{\mu})$  ( $\mathcal{O} \equiv \psi, S, P$ ), as well as their counterparts in  $\overline{MS}$ :  $\Sigma_{\mathcal{O}}^{\overline{MS}}(q, \bar{\mu})$ . The values of all these functions, computed on the lattice, coincide with values computed in dimensional regularization (as can be inferred, e.g., from [1]).

The present work is the first two-loop computation of the renormalization of fermion bilinears on the lattice. One-loop computations of the same quantities exist for quite some time now (see, e.g., [2], [3], [4] and references therein). There have been made several attempts to estimate  $Z_{\mathcal{O}}$  non-perturbatively; recent results can be found in Refs. [5–10]. Some results have also been obtained using stochastic perturbation theory [11]. A related computation, regarding the fermion mass renormalization  $Z_m$  with staggered fermions can be found in [12].

The paper is organized as follows: Section II provides a formulation of the problem, along with all necessary definitions of renormalization schemes and of the quantities to compute. Section III describes our computational methods and the results which are obtained. Finally, in Section IV we discuss some salient features of our calculation, and comment on future extensions to the present work.

Recently, there has been some interest in gauge theories with fermions in representations other than the fundamental. Such theories are being studied in various contexts [13–18], e.g., supersymmetry, phase transitions, and the ‘AdS/QCD’ correspondence. It is relatively straightforward to generalize our results to an arbitrary representation; this is presented in the Appendix.

## II. FORMULATION OF THE PROBLEM

### A. Lattice action

We will make use of the Wilson formulation of the QCD action on the lattice, with the addition of the clover (SW) [19] term for fermions. In standard notation, it reads:

$$\begin{aligned}
S_L = S_G + \sum_f \sum_x (4r + m_o) \bar{\psi}_f(x) \psi_f(x) \\
- \frac{1}{2} \sum_f \sum_{x, \mu} \left[ \bar{\psi}_f(x) (r - \gamma_\mu) U_{x, x+\mu} \psi_f(x + \mu) \right. \\
\left. + \bar{\psi}_f(x + \mu) (r + \gamma_\mu) U_{x+\mu, x} \psi_f(x) \right] \\
+ \frac{i}{4} c_{\text{SW}} \sum_f \sum_{x, \mu, \nu} \bar{\psi}_f(x) \sigma_{\mu\nu} \hat{F}_{\mu\nu}(x) \psi_f(x),
\end{aligned} \tag{1}$$

$$\text{where : } \quad \hat{F}_{\mu\nu} \equiv \frac{1}{8a^2} (Q_{\mu\nu} - Q_{\nu\mu}) \tag{2}$$

$$\begin{aligned}
\text{and : } \quad Q_{\mu\nu} = & U_{x, x+\mu} U_{x+\mu, x+\mu+\nu} U_{x+\mu+\nu, x+\nu} U_{x+\nu, x} \\
& + U_{x, x+\nu} U_{x+\nu, x+\nu-\mu} U_{x+\nu-\mu, x-\mu} U_{x-\mu, x} \\
& + U_{x, x-\mu} U_{x-\mu, x-\mu-\nu} U_{x-\mu-\nu, x-\nu} U_{x-\nu, x} \\
& + U_{x, x-\nu} U_{x-\nu, x-\nu+\mu} U_{x-\nu+\mu, x+\mu} U_{x+\mu, x}
\end{aligned} \tag{3}$$

$S_G$  is the standard pure gluon action, made out of  $1 \times 1$  plaquettes. The clover coefficient  $c_{\text{SW}}$  is treated here as a free parameter;  $r$  is the Wilson parameter (set to  $r = 1$  henceforth);

$f$  is a flavor index;  $\sigma_{\mu\nu} = (i/2)[\gamma_\mu, \gamma_\nu]$ . Powers of the lattice spacing  $a_L$  have been omitted and may be directly reinserted by dimensional counting.

The ‘‘Lagrangian mass’’  $m_o$  is a free parameter here. However, since we will be using mass independent renormalization schemes, all renormalization functions which we will be calculating, must be evaluated at vanishing renormalized mass, that is, when  $m_o$  is set equal to the critical value  $m_{cr}$ :  $m_o \rightarrow m_{cr} = 0 + \mathcal{O}(g_o^2)$ .

## B. Definition of renormalized quantities

As a prerequisite to our programme, we will need the renormalization functions for the gluon, ghost and fermion fields ( $A_\mu^a$ ,  $c^a$ ,  $\psi$ ), and for the coupling constant  $g$  and gauge parameter  $\alpha$ , defined as follows:

$$\begin{aligned} A_{\mu o}^a &= \sqrt{Z_A} A_\mu^a, & c_o^a &= \sqrt{Z_c} c^a, & \psi_o &= \sqrt{Z_\psi} \psi \\ g_o &= \mu^\epsilon Z_g g, & \alpha_o &= Z_a^{-1} Z_A \alpha \end{aligned} \quad (4)$$

The value of each  $Z_O$  depends both on the regularization  $X$  and on the renormalization scheme  $Y$  employed, and thus should properly be denoted as  $Z_O^{X,Y}$ . The scale  $\mu$  enters the relation between  $g_o$  and  $g$  only in dimensional regularization ( $D = 4 - 2\epsilon$  dimensions).

We will need  $Z_A$ ,  $Z_c$ ,  $Z_\alpha$  and  $Z_g$  to 1 loop and  $Z_\psi$  to 2 loops. Our 1-loop results, performed in a generic gauge, are in agreement with results found in the literature (see, e.g., Refs. [4,20]).

## C. Definition of the RI' scheme

This renormalization scheme [21–23] is more immediate for a lattice regularized theory. It is defined by imposing a set of normalization conditions on matrix elements at a scale  $\bar{\mu}$ , where (just as in the  $\overline{MS}$  scheme) [24]:

$$\bar{\mu} = \mu (4\pi/e^{\gamma_E})^{1/2} \quad (5)$$

( $\gamma_E$  is the Euler constant).

In Euclidean space, the fermion self energy  $\Sigma_\psi^L(q, a_L) = i\not{q} + m_o + \mathcal{O}(g_o^2)$  is renormalized through:

$$\lim_{a_L \rightarrow 0} \left[ Z_\psi^{L,RI'}(a_L \bar{\mu}) \text{tr} \left( \Sigma_\psi^L(q, a_L) \not{q} \right) / (4i q^2) \right]_{q^2=\bar{\mu}^2} = 1 \quad (6)$$

The trace here is over Dirac indices; a Kronecker delta in color and in flavor indices has been factored out of the definition of  $\Sigma_\psi^L$ .

Similarly, for the ghost self energy  $\Sigma_c^L(q, a_L) = q^2 + \mathcal{O}(g_o^2)$ :

$$\lim_{a_L \rightarrow 0} \left[ Z_c^{L,RI'}(a_L \bar{\mu}) \frac{\Sigma_c^L(q, a_L)}{q^2} \right]_{q^2=\bar{\mu}^2} = 1 \quad (7)$$

$Z_A$  and  $Z_\alpha$  are extracted from the gluon propagator  $G_{\mu\nu}^L(q, a_L)$  with radiative corrections<sup>1</sup>:

$$G_{\mu\nu}^L(q, a_L) = \frac{1}{q^2} \left[ \frac{\delta_{\mu\nu} - q_\mu q_\nu / q^2}{\Pi_T(a_L q)} + \alpha_o \frac{q_\mu q_\nu / q^2}{\Pi_L(a_L q)} \right] \quad (8)$$

where  $\Pi_{T,L}(a_L q) = 1 + \mathcal{O}(g_o^2)$ . The normalization conditions are:

$$\lim_{a_L \rightarrow 0} \left[ Z_A^{L,RI'}(a_L \bar{\mu}) \frac{1}{\Pi_T(a_L q)} \right]_{q^2=\bar{\mu}^2} = 1 \quad (9)$$

$$\lim_{a_L \rightarrow 0} \left[ Z_\alpha^{L,RI'}(a_L \bar{\mu}) \frac{1}{\Pi_L(a_L q)} \right]_{q^2=\bar{\mu}^2} = 1 \quad (10)$$

We have checked explicitly that  $Z_\alpha^{L,RI'} = 1$  up to one loop, in agreement with the continuum.

For consistency with the Slavnov-Taylor identities,  $Z_g$  in the  $RI'$  scheme is defined as in the  $\overline{MS}$  scheme. In dimensional regularization ( $DR$ ) this is achieved by requiring that the gluon-fermion-antifermion 1PI vertex function,  $G_{A\bar{\psi}\psi}$ , renormalizes as follows [1]:

$$\lim_{\epsilon \rightarrow 0} \left[ Z_\psi^{DR,RI'} (Z_A^{DR,RI'})^{1/2} Z_g^{DR,RI'} G_{A\bar{\psi}\psi}(q) \right]_{q^2=\bar{\mu}^2} = G_{A\bar{\psi}\psi}^{\text{finite}} \quad (11)$$

The value of  $Z_g$  is tuned in such a way as to absorb only the poles in  $\epsilon$  which appear in  $G_{A\bar{\psi}\psi}$  (together with matching powers of  $\ln(4\pi) - \gamma_E$ ); this leads to a result for  $G_{A\bar{\psi}\psi}^{\text{finite}}$  which is finite but not unity. Before rescaling, we have first divided  $G_{A\bar{\psi}\psi}$  by the bare coupling constant, as in Ref. [25], in order to have unity as the tree level value for  $G_{A\bar{\psi}\psi}^{\text{finite}}$ . We have set the fermion momentum to zero;  $q$  refers to the gluon/antifermion momentum. Alternatively, a similar procedure can be performed on the gluon-ghost-antighost vertex:

$$\lim_{\epsilon \rightarrow 0} \left[ Z_c^{DR,RI'} (Z_A^{DR,RI'})^{1/2} Z_g^{DR,RI'} G_{A\bar{c}c}(q) \right]_{q^2=\bar{\mu}^2} = G_{A\bar{c}c}^{\text{finite}} \quad (12)$$

Eq.(12) leads to exactly the same value for  $Z_g$ .

The corresponding renormalization conditions on the lattice read:

$$\lim_{a_L \rightarrow 0} \left[ Z_\psi^{L,RI'} (Z_A^{L,RI'})^{1/2} Z_g^{L,RI'} G_{A\bar{\psi}\psi}^L(q, a_L) \right]_{q^2=\bar{\mu}^2} = G_{A\bar{\psi}\psi}^{\text{finite}} \quad (13)$$

or, equivalently:

$$\lim_{a_L \rightarrow 0} \left[ Z_c^{L,RI'} (Z_A^{L,RI'})^{1/2} Z_g^{L,RI'} G_{A\bar{c}c}^L(q, a_L) \right]_{q^2=\bar{\mu}^2} = G_{A\bar{c}c}^{\text{finite}} \quad (14)$$

where the expressions  $G_{A\bar{\psi}\psi}^{\text{finite}}$  and  $G_{A\bar{c}c}^{\text{finite}}$  are required to be the *same* as those stemming from the continuum (Eqs.(11),(12)). We have calculated  $Z_g^{L,RI'}$ , using either one of Eqs.(13), (14), and have verified that the same result is obtained.

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<sup>1</sup>One should carefully distinguish among the following standard symbols:  $a_L$ : lattice spacing;  $\alpha_o$ ,  $\alpha_{RI'}$ ,  $\alpha_{\overline{MS}}$ : bare and renormalized gauge parameters.

## D. Conversion to the $\overline{MS}$ scheme

For easier comparison with calculations coming from the continuum, we need to express our results in the  $\overline{MS}$  scheme. Each renormalization function on the lattice,  $Z_O^{L,RI'}$ , may be expressed as a power series in the renormalized coupling constant  $g_{RI'}$ . For the purposes of our work the conversion of  $g_{RI'}$  to  $\overline{MS}$  is trivial since:

$$g_{RI'} = g_{\overline{MS}} + \mathcal{O}(g_{\overline{MS}}^9) \quad (15)$$

As already mentioned, our 1-loop calculations for  $Z_A$ ,  $Z_c$ ,  $Z_\alpha$  and  $Z_g$  are performed in a generic gauge,  $\alpha_{RI'}$ . The conversion to the  $\overline{MS}$  scheme is given by [26]:

$$\alpha_{RI'} = \frac{Z_A^{L,\overline{MS}}}{Z_A^{L,RI'}} \alpha_{\overline{MS}} \equiv \alpha_{\overline{MS}} / C_A(g_{\overline{MS}}, \alpha_{\overline{MS}}) \quad (16)$$

Since the ratio of  $Z$ 's appearing in Eq.(16) must be *regularization independent*, it may be calculated more easily in dimensional regularization [1]; to 1 loop, the conversion factor  $C_A$  equals:

$$C_A(g, \alpha) = \frac{Z_A^{DR,RI'}}{Z_A^{DR,\overline{MS}}} = 1 + \frac{g^2}{36(16\pi^2)} \left[ (9\alpha^2 + 18\alpha + 97) N_c - 40N_f \right] \quad (17)$$

(Here, and throughout the rest of this work, both  $g$  and  $\alpha$  are in the  $\overline{MS}$  scheme, unless specified otherwise.)

Once we have computed the renormalization functions in the  $RI'$  scheme we can construct their  $\overline{MS}$  counterparts using conversion factors which, up to the required perturbative order, are given by:

$$C_c(g, \alpha) \equiv \frac{Z_c^{L,RI'}}{Z_c^{L,\overline{MS}}} = \frac{Z_c^{DR,RI'}}{Z_c^{DR,\overline{MS}}} = 1 + \frac{g^2}{16\pi^2} N_c \quad (18)$$

$$\begin{aligned} C_\psi(g, \alpha) &\equiv \frac{Z_\psi^{L,RI'}}{Z_\psi^{L,\overline{MS}}} = \frac{Z_\psi^{DR,RI'}}{Z_\psi^{DR,\overline{MS}}} \\ &= 1 - \frac{g^2}{16\pi^2} c_F \alpha + \frac{g^4}{8(16\pi^2)^2} c_F \left[ (8\alpha^2 + 5) c_F + 14 N_f \right. \\ &\quad \left. - (9\alpha^2 - 24\zeta(3) \alpha + 52\alpha - 24\zeta(3) + 82) N_c \right] \end{aligned} \quad (19)$$

where  $c_F = (N_c^2 - 1)/(2 N_c)$  is the quadratic Casimir operator in the fundamental representation of the color group;  $\zeta(x)$  is Riemann's zeta function. (We employ a standard normalization for the generators of the algebra,  $T^a$ , see the Appendix.)

## E. Renormalization of fermion bilinears

The lattice operators  $\mathcal{O}_\Gamma = \bar{\psi} \Gamma \psi$  must, in general, be renormalized in order to have finite matrix elements. We define renormalized operators by

$$\mathcal{O}_\Gamma^{RI'} = Z_\Gamma^{L,RI'}(a_L \bar{\mu}) \mathcal{O}_{\Gamma_0} \quad (20)$$

The flavor singlet scalar operator receives also an additive renormalization, which must be taken into account; we discuss this issue in the following subsection. For the scalar (S) and pseudoscalar (P) operators, the renormalization functions  $Z_\Gamma^{L,RI'}$  can be obtained through the corresponding bare 2-point functions  $\Sigma_\Gamma^L(qa_L)$  (amputated, 1PI) on the lattice, in the following way:

$$\lim_{a_L \rightarrow 0} \left[ Z_\psi^{L,RI'} Z_S^{L,RI'} \Sigma_S^L(qa_L) \right]_{q^2=\bar{\mu}^2} = \mathbb{1} \quad (21)$$

$$\lim_{a_L \rightarrow 0} \left[ Z_\psi^{L,RI'} Z_P^{L,RI'} \Sigma_P^L(qa_L) \right]_{q^2=\bar{\mu}^2} = \gamma_5 \quad (22)$$

where:

$$\Sigma_S^L(qa_L) = \mathbb{1} + \mathcal{O}(g_o^2) \quad , \quad \Sigma_P^L(qa_L) = \gamma_5 + \mathcal{O}(g_o^2) \quad (23)$$

Once the quantities  $Z_\Gamma^{L,RI'}$  have been calculated, one may proceed to compute them also in the  $\overline{MS}$  scheme. In the case of the scalar operator ( $\mathcal{O}_{S_0} = \bar{\psi}_o \psi_o$ ), the renormalization function,  $Z_S^{L,\overline{MS}}$ , can be obtained by:

$$Z_S^{L,\overline{MS}} = Z_S^{L,RI'} / C_S(g, \alpha) \quad (24)$$

where  $C_S(g, \alpha)$  is a *regularization independent* conversion factor and has been calculated in dimensional regularization [1]:

$$\begin{aligned} C_S(g, \alpha) &\equiv \frac{Z_S^{L,RI'}}{Z_S^{L,\overline{MS}}} = \frac{Z_S^{DR,RI'}}{Z_S^{DR,\overline{MS}}} \\ &= 1 + \frac{g^2}{16\pi^2} c_F (\alpha + 4) + \frac{g^4}{24(16\pi^2)^2} c_F \left[ (24\alpha^2 + 96\alpha - 288\zeta(3) + 57) c_F + 166 N_f \right. \\ &\quad \left. - (18\alpha^2 + 84\alpha - 432\zeta(3) + 1285) N_c \right] \end{aligned} \quad (25)$$

The treatment of the pseudoscalar operator ( $\mathcal{O}_{P_0} = \bar{\psi}_o \gamma_5 \psi_o$ ) in the  $\overline{MS}$  scheme requires special attention, due to the non-unique generalization of  $\gamma_5$  to D dimensions. A practical definition of  $\gamma_5$  for multiloop calculations, which is most commonly employed in dimensional regularization and does not suffer from inconsistencies is [27]:

$$\gamma_5 = i \frac{1}{4!} \epsilon_{\nu_1 \nu_2 \nu_3 \nu_4} \gamma_{\nu_1} \gamma_{\nu_2} \gamma_{\nu_3} \gamma_{\nu_4} \quad , \quad \nu_i = 0, 1, 2, 3 \quad (26)$$

Of course,  $\gamma_5$  as defined in Eq.(26) does not anticommute with the D-dimensional  $\gamma_\mu$ ; an ultimate consequence of this fact is that Ward identities involving the axial and pseudoscalar operators, renormalized in this way, are violated.

To obtain the correctly renormalized pseudoscalar operator, one must introduce an extra *finite* factor,  $Z_5$ , in addition to the usual renormalization function  $Z_P^{DR,\overline{MS}}$  which only contains poles in  $\epsilon$ . We set:

$$\mathcal{O}_P = Z_5(g) Z_P^{DR, \overline{MS}} \mathcal{O}_{P_0} \quad (27)$$

$Z_5$  is defined by the requirement that the scalar and pseudoscalar renormalized Green's functions coincide:

$$Z_5 \equiv \frac{G_S^{\overline{MS}} \gamma_5}{G_P^{\overline{MS}}} \quad (28)$$

The value of  $Z_5$ , calculated in Ref. [28], is gauge independent; it equals:

$$Z_5(g) = 1 - \frac{g^2}{16\pi^2} (8 c_F) + \frac{g^4}{(16\pi^2)^2} \left( \frac{2}{9} c_F N_c + \frac{4}{9} c_F N_f \right) + \mathcal{O}(g^6) \quad (29)$$

$Z_P^{L, \overline{MS}}$  can now be obtained by:

$$Z_P^{L, \overline{MS}} = Z_P^{L, RI'} / (C_S Z_5) \quad (30)$$

Similarly, one can convert the  $RI'$  renormalized Green's functions,  $G_{\Gamma}^{RI'}$ , to their  $\overline{MS}$  counterparts, through:

$$\frac{G_S^{RI'}}{G_S^{\overline{MS}}} = C_\psi C_S \quad , \quad \frac{G_P^{RI'}}{G_P^{\overline{MS}}} = C_\psi C_S Z_5 \quad (31)$$

(In Eqs.(30, 31) it is understood that powers of  $g_{RI'}$ ,  $\alpha_{RI'}$ , implicit in  $RI'$  quantities, must also be converted to  $g_{\overline{MS}}$ ,  $\alpha_{\overline{MS}}$ , respectively, using Eqs.(15, 16). )

## F. Fermion Mass Renormalization

As a by-product of this work, one can evaluate the fermion multiplicative mass renormalization,  $Z_m$ , which is directly related to the scalar flavor singlet operator. This operator differs from the ones considered thus far, in that it receives also an additive renormalization, since it has a nonzero perturbative vacuum expectation value; thus, it mixes with the identity at the quantum level. Once its vacuum expectation value is subtracted, the resulting operator is multiplicative renormalizable. The renormalization is then simply given by only connected diagrams of the original operator (Figs. 3, 4 and 5); all disconnected diagrams are easily shown to cancel out.

The perturbative vacuum expectation value<sup>2</sup> is of course a power divergent quantity, and it cannot be expected to approach well the value of the corresponding disconnected matrix elements in numerical simulations. Fortunately, this quantity is not needed for multiplicative renormalization, as mentioned above. However, as regards simulations, one should bear in mind that disconnected parts must be evaluated and subtracted from matrix elements, before the latter can be renormalized.

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<sup>2</sup>For a tree level computation of this quantity, see Ref. [29].



Let us express the fermion self energy in the following way:

$$\Sigma_\psi^{L,RI'} = i\not{q} \Sigma_{\text{odd}}(qa_L, m_o a_L, g_o) + \mathbb{1} \cdot \frac{1}{a_L} \Sigma_{\text{even}}(qa_L, m_o a_L, g_o) \quad (32)$$

where  $\Sigma_{\text{odd}} = 1 + \mathcal{O}(g_o^2)$  and  $\Sigma_{\text{even}} = m_o a_L + \mathcal{O}(g_o^2)$ . Terms like  $\sum_\mu q_\mu^3 \gamma_\mu / q^2$ , though a priori allowed by hypercubic symmetry, are eventually seen to cancel, as expected by Lorentz invariance.

For generic values of  $m_o$ , the even part of  $\Sigma_\psi^{L,RI'}$  is power divergent; in order to achieve a finite renormalized mass,  $m_r$ , the values of the Lagrangian mass  $m_o$  must be near a critical value,  $m_{\text{cr}}$ , at which  $\Sigma_{\text{even}}$  vanishes:  $\Sigma_{\text{even}}(qa_L, m_{\text{cr}} a_L, g_o) = 0 + \mathcal{O}(q^2 a_L^2)$ . That is,  $m_{\text{cr}}$  is required to satisfy:

$$\Sigma_{\text{even}}(0, m_{\text{cr}} a_L, g_o) = 0 \quad (33)$$

This is a recursive equation which can be solved for  $m_{\text{cr}}$  order-by-order in perturbation theory. Its value is known to two loops for Wilson fermions: [30] (confirmed independently in [31]), and for clover fermions: [32] (with Wilson gluons), [33] (with Symanzik gluons). Only the 1-loop value of  $m_{\text{cr}}$  enters the present calculation.

We can perform a Taylor expansion with respect to the bare mass<sup>3</sup>,  $m_B \equiv m_o - m_{\text{cr}}$ , for both  $\Sigma_{\text{odd}}$  and  $\Sigma_{\text{even}}$ :

$$\Sigma_{\text{odd}}(qa_L, m_o a_L, g_o) = \left[ \Sigma_{\text{odd}}(qa_L, m_o a_L, g_o) \right]_{m_o=m_{\text{cr}}} + \mathcal{O}(m_B a_L) \quad (34)$$

$$\begin{aligned} \frac{1}{a_L} \Sigma_{\text{even}}(qa_L, m_o a_L, g_o) &= \frac{1}{a_L} \left[ \Sigma_{\text{even}}(qa_L, m_o a_L, g_o) \right]_{m_o=m_{\text{cr}}} \\ &+ m_B \left[ \frac{\partial}{\partial (m_o a_L)} \Sigma_{\text{even}}(qa_L, m_o a_L, g_o) \right]_{m_o=m_{\text{cr}}} + \mathcal{O}(a_L) \end{aligned} \quad (35)$$

Note that when  $m_o = m_{\text{cr}}$ , the first term on the r.h.s. of Eq.(35) vanishes in the limit  $a_L \rightarrow 0$ , by virtue of Eq.(33).

Having in mind that, in calculating  $\Sigma_\psi^{L,RI'}$ , one is interested in the limit  $a_L \rightarrow 0$ , the fermion self energy takes the form:

$$\begin{aligned} \Sigma_\psi^{L,RI'} &= i\not{q} \left[ \Sigma_{\text{odd}}(qa_L, m_o a_L, g_o) \right]_{m_o=m_{\text{cr}}} \\ &+ \mathbb{1} \cdot m_B \left[ \frac{\partial}{\partial (m_o a_L)} \Sigma_{\text{even}}(qa_L, m_o a_L, g_o) \right]_{m_o=m_{\text{cr}}} \end{aligned} \quad (36)$$

The renormalized fermion mass is now defined by:

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<sup>3</sup>Note that  $m_{\text{cr}}$  (and, consequently,  $m_o$ ) is power divergent in  $a_L$  since its calculation contains no other dimensional quantities;  $m_B$ , on the other hand is at most logarithmically divergent in  $a_L$ .

$$m_r = \left( Z_m^{L,RI'} \right)^{-1} m_B \quad (37)$$

The renormalization condition for  $Z_\psi^{L,RI'}$  (Eq.(6)) for nonzero  $m_r$ , reads:

$$\lim_{a_L \rightarrow 0} \left[ Z_\psi^{L,RI'} \Sigma_\psi^{L,RI'} - (i\not{q} + m_r) \right]_{q^2=\bar{\mu}^2} = 0 \quad (38)$$

By combining Eqs.(37) and (38) we find the renormalization condition for  $Z_m^{L,RI'}$ :

$$\lim_{a_L \rightarrow 0} \left[ Z_\psi^{L,RI'} Z_m^{L,RI'} \left[ \frac{\partial}{\partial (m_o a_L)} \Sigma_{\text{even}}(q a_L, m_o a_L, g_o) \right]_{m_o=m_{\text{cr}}} \right]_{q^2=\bar{\mu}^2} = 1 \quad (39)$$

We stress again that, even though the Lagrangian mass  $m_o$  may take arbitrary values, the renormalization condition involves only  $m_o \rightarrow m_{\text{cr}}$ .

In order to establish a relation between  $Z_m^{L,RI'}$  and  $Z_{S,\text{singlet}}^{L,RI'}$ , note that Eq.(39) coincides with Eq.(21) if  $\partial/\partial(m_o a_L) \Sigma_{\text{even}} = \Sigma_{S,\text{singlet}}^{L,RI'}$ . Indeed, the equality between  $\partial/\partial(m_o a_L) \Sigma_{\text{even}}$  and  $\Sigma_{S,\text{singlet}}^{L,RI'}$  holds diagram by diagram in perturbation theory, noting that:

- The tree level value equals 1, in both cases
- The effect of inserting the scalar operator on a given fermion propagator of any self-energy Feynman diagram is equivalent to taking the negative partial derivative  $-\partial/\partial(m_o a_L)$  of that propagator
- Combinatorial factors agree
- There is an extra minus sign in the geometric series summation of 1PI diagrams leading to the fermion self-energy

Once all of the above statements are taken into account, one comes to the conclusion that:

$$Z_m^{L,RI'} = Z_{S,\text{singlet}}^{L,RI'} \quad (40)$$

Given that  $m_{\text{cr}} a_L = \mathcal{O}(g_o^2)$ , all two-loop calculations can be performed with strictly massless fermion propagators, provided that appropriate fermion mass counterterms are introduced on one-loop diagrams.

### III. COMPUTATION AND RESULTS

The Feynman diagrams relevant to the fermion self-energy  $\Sigma_\psi^L(q, a_L)$ , at 1- and 2-loop level, are shown in Figs.1 and 2, respectively; those relevant to  $\Sigma_S^L(q a_L)$ ,  $\Sigma_P^L(q a_L)$  are shown in Figs.3 and 4.

For flavor singlet bilinears, there are 4 extra diagrams, in addition to those of Fig.4, shown in Fig.5; in these diagrams, the operator insertion occurs inside a closed fermion loop.

The evaluation and algebraic manipulation of Feynman diagrams, leading to a code for numerical loop integration, is performed automatically using our software for Lattice Perturbation Theory, written in Mathematica.

The most laborious aspect of the procedure is the extraction of the dependence on the external momentum  $q$ . This is a delicate task at two loops; for this purpose, we cast algebraic expressions (typically involving thousands of summands) into terms which can be naively Taylor expanded in  $q$  to the required order, plus a smaller set of terms containing superficial divergences and/or subdivergences. The latter can be evaluated by an extension of the method of Ref. [34] to 2 loops; this entails analytical continuation to  $D > 4$  dimensions, and splitting each expression into a UV-finite part (which can thus be calculated in the continuum, using the methods of Ref. [35]), and a part which is polynomial in  $q$ . A primitive set of divergent lattice integrals involving gluon propagators, which can be obtained in this manner, can be found in Ref. [36].

Some of the diagrams contributing to  $\Sigma_\psi^L(q, a_L)$ ,  $\Sigma_S^L(qa_L)$  and  $\Sigma_P^L(qa_L)$  are infrared divergent when considered separately, and thus must be grouped together in order to give finite results. Such groups are formed by diagrams (7-11), (12-13), (14-18), (19-20), (21-23) in Fig.2, diagrams (3-7), (8-9), (10-11,19) in Fig.4 and diagrams (1-2), (3-4) in Fig.5.

In Figures 1-5, “mirror” diagrams (those in which the direction of the external fermion line is reversed) should also be taken into account. In most cases, these coincide trivially with the original diagrams; even in the remaining cases, they can be seen to give equal contribution, by invariance under charge conjugation.

As mentioned before, all calculations should be performed at vanishing renormalized mass; this can be achieved by working with massless fermion propagators, provided an appropriate fermion mass counterterm is introduced (diagram 23 in Fig.2 and diagram 11 in Fig.4).

All two-loop diagrams have been calculated in the bare Feynman gauge ( $\alpha_o = 1$ ). One-loop diagrams have been calculated for generic values of  $\alpha_o$ ; this allows us to convert our two-loop results to the renormalized Feynman gauge ( $\alpha_{RI'} = 1$  or  $\alpha_{\overline{MS}} = 1$ ).

Numerical loop integration was carried out by our “integrator” program, a *metacode* written in Mathematica, for converting lengthy integrands into efficient Fortran code. Two-loop numerical integrals were evaluated for lattices of size up to  $L = 40$ ; the results were then extrapolated to  $L \rightarrow \infty$ . Extrapolation is the only source of systematic error; this error can be estimated quite accurately (see, e.g. Ref. [37]), given that  $L$ -dependence of results can only span a restricted set of functional forms.



FIG. 1. One-loop diagrams contributing to  $Z_\psi$ .  
A wavy (solid) line represents gluons (fermions).

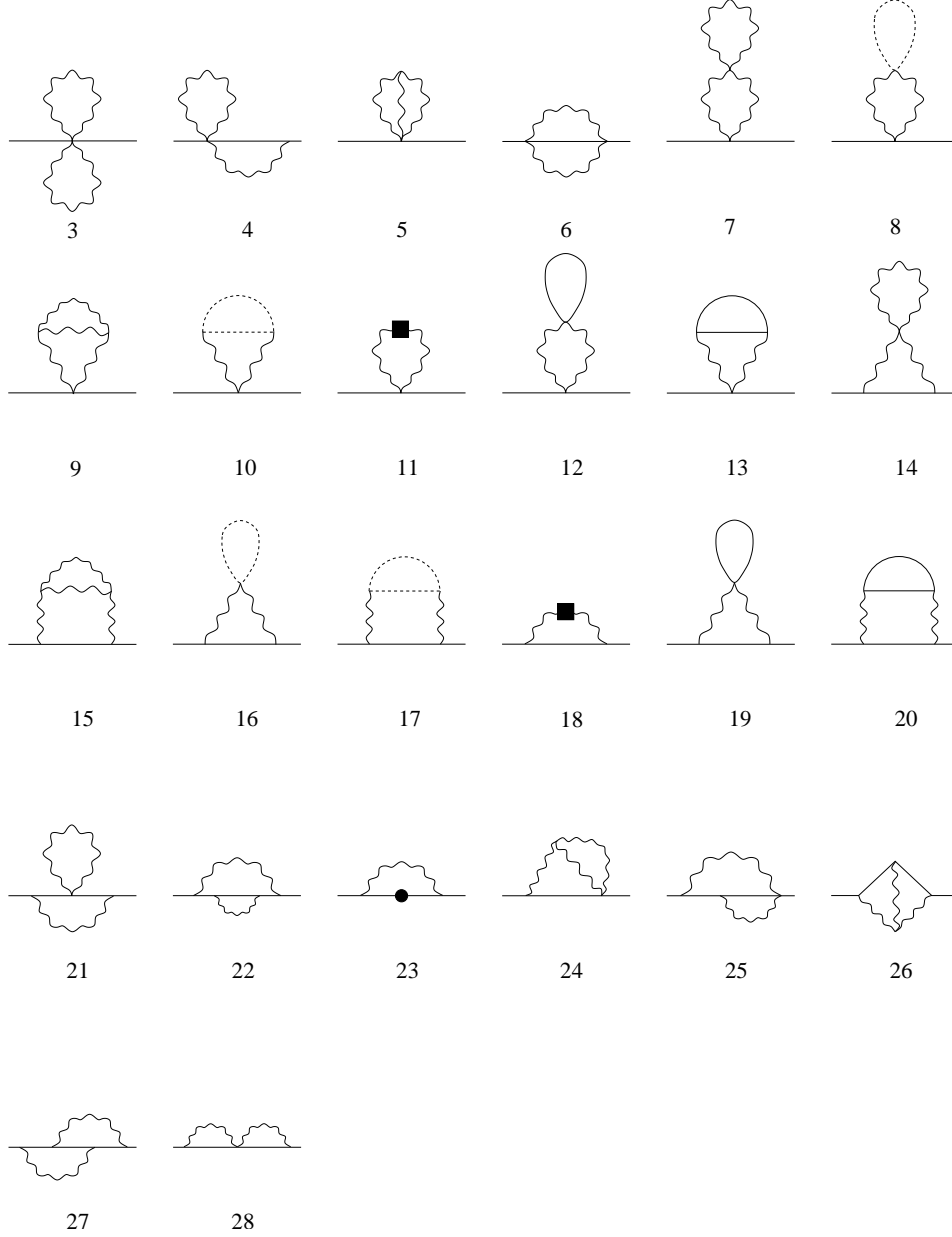


FIG. 2. Two-loop diagrams contributing to  $Z_\psi$ . Wavy (solid, dotted) lines represent gluons (fermions, ghosts). Solid boxes denote vertices stemming from the measure part of the action; a solid circle is a fermion mass counterterm.

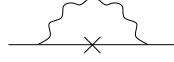


FIG. 3. One-loop diagram contributing to  $Z_S$  and  $Z_P$ . A wavy (solid) line represents gluons (fermions). A cross denotes the Dirac matrices  $\mathbb{1}$  (scalar) and  $\gamma_5$  (pseudoscalar).

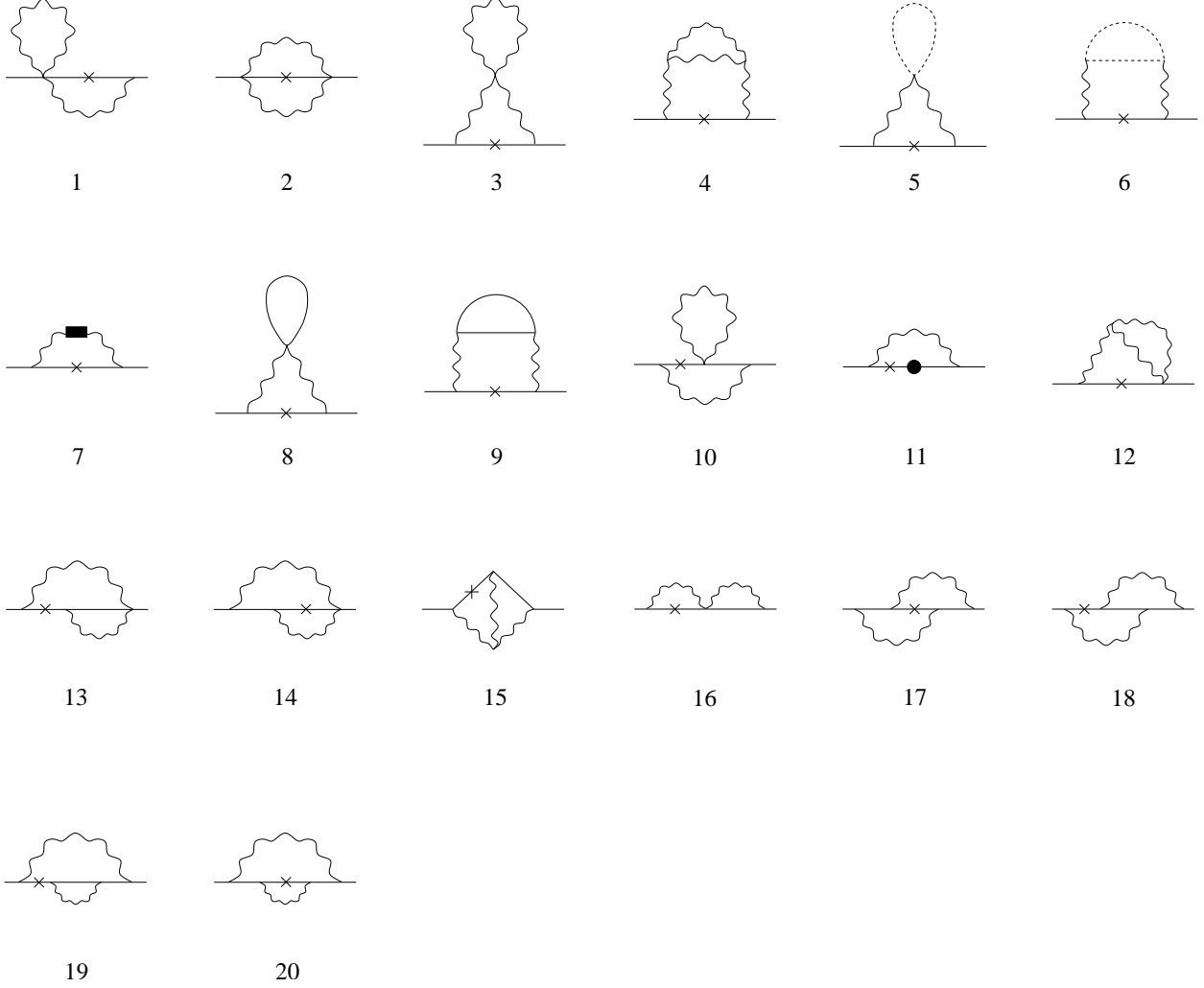


FIG. 4. Two-loop diagrams contributing to  $Z_S$  and  $Z_P$ . Wavy (solid, dotted) lines represent gluons (fermions, ghosts). A solid box denotes a vertex from the measure part of the action; a solid circle is a mass counterterm; crosses denote the matrices  $\mathbb{1}$  (scalar) and  $\gamma_5$  (pseudoscalar).

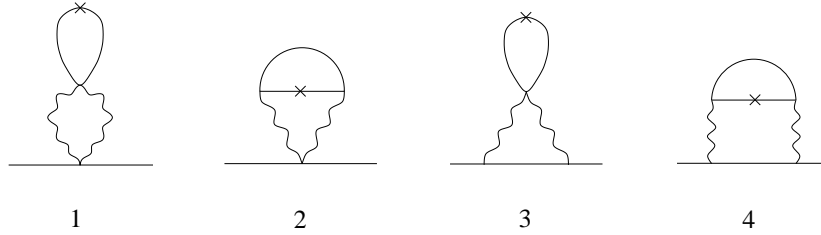


FIG. 5. Extra two-loop diagrams contributing to  $Z_{S,singlet}$ . A cross denotes an insertion of a flavor singlet operator. Wavy (solid) lines represent gluons (fermions).

## A. One-loop results

1-loop results for  $Z_\psi^{L,RI'}$ ,  $Z_S^{L,RI'}$  and  $Z_P^{L,RI'}$  are presented below in a generic gauge. The errors result from the  $L \rightarrow \infty$  extrapolation.

$$Z_\psi^{L,RI'} = 1 + \frac{g_o^2}{16\pi^2} c_F \left[ \left( \ln(a_L^2 \bar{\mu}^2) - 4.792009570(1) \right) \alpha_o + 16.644413858(5) \right. \\ \left. - 2.248868528(3) c_{\text{SW}} - 1.397267102(5) c_{\text{SW}}^2 \right] \quad (41)$$

$$Z_S^{L,RI'} = 1 + \frac{g_o^2}{16\pi^2} c_F \left[ 3 \ln(a_L^2 \bar{\mu}^2) - \alpha_o - 16.9524103(1) \right. \\ \left. - 7.7379159(3) c_{\text{SW}} + 1.38038065(4) c_{\text{SW}}^2 \right] \quad (42)$$

$$Z_P^{L,RI'} = 1 + \frac{g_o^2}{16\pi^2} c_F \left[ 3 \ln(a_L^2 \bar{\mu}^2) - \alpha_o - 26.5954414(1) \right. \\ \left. + 2.248868528(3) c_{\text{SW}} - 2.03601561(4) c_{\text{SW}}^2 \right] \quad (43)$$

The corresponding quantities in the  $\overline{MS}$  scheme are:

$$Z_\psi^{L,\overline{MS}} = 1 + \frac{g_o^2}{16\pi^2} c_F \left[ \left( \ln(a_L^2 \bar{\mu}^2) - 3.792009570(1) \right) \alpha_o + 16.644413858(5) \right. \\ \left. - 2.248868528(3) c_{\text{SW}} - 1.397267102(5) c_{\text{SW}}^2 \right] \quad (44)$$

$$Z_S^{L,\overline{MS}} = 1 + \frac{g_o^2}{16\pi^2} c_F \left[ 3 \ln(a_L^2 \bar{\mu}^2) - 12.9524103(1) \right. \\ \left. - 7.7379159(3) c_{\text{SW}} + 1.38038065(4) c_{\text{SW}}^2 \right] \quad (45)$$

$$Z_P^{L,\overline{MS}} = 1 + \frac{g_o^2}{16\pi^2} c_F \left[ 3 \ln(a_L^2 \bar{\mu}^2) - 14.5954414(1) \right. \\ \left. + 2.248868528(3) c_{\text{SW}} - 2.03601561(4) c_{\text{SW}}^2 \right] \quad (46)$$

Our results confirm the existing results found in the literature [4] (note, however, a difference in  $Z_P^{L,\overline{MS}}$ ; this is entirely due to the factor  $Z_5$  in Eq.(30)).

## B. Two-loop results

The evaluation of all Feynman diagrams in Figs.1-5 leads directly to the corresponding bare Green's functions  $\Sigma_\psi^L$ ,  $\Sigma_S^L$  and  $\Sigma_P^L$ . These, in turn, can be converted to the corresponding renormalization functions  $Z_\psi^{L,Y}$ ,  $Z_S^{L,Y}$  and  $Z_P^{L,Y}$  ( $Y = RI'$  or  $\overline{MS}$ ), via Eqs.(6), (21) and (22). To this end, we need the following one-loop expression for  $Z_A^{L,Y}$  (recall that  $Z_\alpha = 1$  to this order):

$$\begin{aligned} Z_A^{L,RI'} &= Z_A^{L,\overline{MS}} + \mathcal{O}(g_o^4) \\ &= 1 + \frac{g_o^2}{16\pi^2} \left[ \ln(a_L^2 \bar{\mu}^2) \left( \frac{2}{3} N_f - \frac{5}{3} N_c \right) \right. \\ &\quad \left. + N_f \left( -2.168501047(1) + 0.7969452308(4) c_{\text{SW}} - 4.7126914428(1) c_{\text{SW}}^2 \right) \right. \\ &\quad \left. + 39.47841760436(1) c_F + 1.94017130069(1) N_c \right] + \mathcal{O}(g_o^4) \end{aligned} \quad (47)$$

To express our results in terms of the renormalized coupling constant, we also need the one-loop expression for  $Z_g^{L,Y}$ :

$$\begin{aligned} Z_g^{L,RI'} &= Z_g^{L,\overline{MS}} + \mathcal{O}(g_o^4) \\ &= 1 + \frac{g_o^2}{16\pi^2} \left[ \ln(a_L^2 \bar{\mu}^2) \left( -\frac{1}{3} N_f + \frac{11}{6} N_c \right) \right. \\ &\quad \left. + N_f \left( 0.5286949677(5) - 0.3984726154(2) c_{\text{SW}} + 2.35634572140(7) c_{\text{SW}}^2 \right) \right. \\ &\quad \left. - 19.73920880218(1) c_F - 3.54958342046(1) N_c \right] + \mathcal{O}(g_o^4) \end{aligned} \quad (48)$$

Eqs.(47, 48) are in agreement with older references (see, e.g., Ref. [20]).

We present below  $Z_\psi^{L,RI'}$ ,  $Z_S^{L,RI'}$  and  $Z_P^{L,RI'}$  to two loops in the renormalized Feynman gauge  $\alpha_{RI'} = 1$ ; we also present the  $\overline{MS}$  analogues  $Z_\psi^{L,\overline{MS}}$ ,  $Z_S^{L,\overline{MS}}$  and  $Z_P^{L,\overline{MS}}$  in the gauge  $\alpha_{\overline{MS}} = 1$ . For conciseness, we omit the bare Green's functions; it is a straightforward exercise to recover these from the corresponding  $Z$ 's.

$$\begin{aligned}
Z_{\psi}^{L,RI'} = 1 &+ \frac{g_o^2}{16\pi^2} c_F \left[ \ln(a_L^2 \bar{\mu}^2) + 11.852404288(5) - 2.248868528(3) c_{\text{SW}} - 1.397267102(5) c_{\text{SW}}^2 \right] \\
&+ \frac{g_o^4}{(16\pi^2)^2} c_F \left[ \ln^2(a_L^2 \bar{\mu}^2) \left( \frac{1}{2} c_F + \frac{2}{3} N_f - \frac{8}{3} N_c \right) \right. \\
&\quad + \ln(a_L^2 \bar{\mu}^2) \left( -6.36317446(8) N_f + 0.79694523(2) N_f c_{\text{SW}} \right. \\
&\quad \quad - 4.712691443(4) N_f c_{\text{SW}}^2 \\
&\quad \quad + 49.83082185(5) c_F - 2.24886861(7) c_F c_{\text{SW}} \\
&\quad \quad \left. \left. - 1.39726705(1) c_F c_{\text{SW}}^2 + 29.03029398(4) N_c \right) \right. \\
&\quad + N_f \left( -7.838(2) + 1.153(1) c_{\text{SW}} + 3.202(3) c_{\text{SW}}^2 \right. \\
&\quad \quad \left. + 6.2477(6) c_{\text{SW}}^3 + 4.0232(6) c_{\text{SW}}^4 \right) \\
&\quad + c_F \left( 505.39(1) - 58.210(9) c_{\text{SW}} + 20.405(5) c_{\text{SW}}^2 \right. \\
&\quad \quad \left. + 18.8431(8) c_{\text{SW}}^3 + 4.2793(2) c_{\text{SW}}^4 \right) \\
&\quad + N_c \left( -20.59(1) - 3.190(5) c_{\text{SW}} - 23.107(6) c_{\text{SW}}^2 \right. \\
&\quad \quad \left. \left. - 5.7234(5) c_{\text{SW}}^3 - 0.7938(1) c_{\text{SW}}^4 \right) \right] \tag{49}
\end{aligned}$$

$$\begin{aligned}
Z_{\psi}^{L,\overline{MS}} = 1 &+ \frac{g_o^2}{16\pi^2} c_F \left[ \ln(a_L^2 \bar{\mu}^2) + 12.852404288(5) - 2.248868528(3) c_{\text{SW}} - 1.397267102(5) c_{\text{SW}}^2 \right] \\
&+ \frac{g_o^4}{(16\pi^2)^2} c_F \left[ \ln^2(a_L^2 \bar{\mu}^2) \left( \frac{2}{3} N_f + \frac{1}{2} c_F - \frac{8}{3} N_c \right) \right. \\
&\quad + \ln(a_L^2 \bar{\mu}^2) \left( -4.58539668(8) N_f + 0.79694523(2) N_f c_{\text{SW}} \right. \\
&\quad \quad - 4.712691443(4) N_f c_{\text{SW}}^2 \\
&\quad \quad + 50.83082185(5) c_F - 2.24886861(7) c_F c_{\text{SW}} \\
&\quad \quad \left. \left. - 1.39726705(1) c_F c_{\text{SW}}^2 + 21.91918287(4) N_c \right) \right. \\
&\quad + N_f \left( -15.970(2) + 1.950(1) c_{\text{SW}} - 1.510(3) c_{\text{SW}}^2 \right. \\
&\quad \quad \left. + 6.2477(6) c_{\text{SW}}^3 + 4.0232(6) c_{\text{SW}}^4 \right) \\
&\quad + c_F \left( 556.10(1) - 60.459(9) c_{\text{SW}} + 19.007(5) c_{\text{SW}}^2 \right. \\
&\quad \quad \left. + 18.8431(8) c_{\text{SW}}^3 + 4.2793(2) c_{\text{SW}}^4 \right) \\
&\quad + N_c \left( 13.68(1) - 3.190(5) c_{\text{SW}} - 23.107(6) c_{\text{SW}}^2 \right. \\
&\quad \quad \left. \left. - 5.7234(5) c_{\text{SW}}^3 - 0.7938(1) c_{\text{SW}}^4 \right) \right] \tag{50}
\end{aligned}$$



$$\begin{aligned}
Z_S^{L,RI'} = 1 &+ \frac{g_o^2}{16\pi^2} c_F \left[ 3 \ln(a_L^2 \bar{\mu}^2) - 17.9524103(1) - 7.7379159(3) c_{\text{SW}} + 1.38038065(4) c_{\text{SW}}^2 \right] \\
&+ \frac{g_o^4}{(16\pi^2)^2} c_F \left[ \ln^2(a_L^2 \bar{\mu}^2) \left( \frac{9}{2} c_F + N_f - \frac{11}{2} N_c \right) \right. \\
&\quad + \ln(a_L^2 \bar{\mu}^2) \left( -8.1721694(5) N_f + 2.3908354(3) N_f c_{\text{SW}} \right. \\
&\quad \quad - 14.13807433(4) N_f c_{\text{SW}}^2 \\
&\quad \quad + 66.0780218(9) c_F - 23.213749(2) c_F c_{\text{SW}} \\
&\quad \quad \left. + 4.1411425(3) c_F c_{\text{SW}}^2 + 55.7975008(9) N_c \right) \\
&\quad + N_f \left( 24.003(3) + 11.878(5) c_{\text{SW}} + 25.59(1) c_{\text{SW}}^2 \right. \\
&\quad \quad \left. + 22.078(3) c_{\text{SW}}^3 - 6.1807(8) c_{\text{SW}}^4 \right) \\
&\quad + c_F \left( -602.35(6) + 91.07(7) c_{\text{SW}} + 51.15(5) c_{\text{SW}}^2 \right. \\
&\quad \quad \left. - 27.759(4) c_{\text{SW}}^3 - 2.688(1) c_{\text{SW}}^4 \right) \\
&\quad + N_c \left( -38.16(4) - 132.40(5) c_{\text{SW}} - 4.04(3) c_{\text{SW}}^2 \right. \\
&\quad \quad \left. + 12.576(3) c_{\text{SW}}^3 + 1.0175(8) c_{\text{SW}}^4 \right) \Big] \tag{51}
\end{aligned}$$

$$\begin{aligned}
Z_S^{L,\overline{MS}} = 1 &+ \frac{g_o^2}{16\pi^2} c_F \left[ 3 \ln(a_L^2 \bar{\mu}^2) - 12.9524103(1) - 7.7379159(3) c_{\text{SW}} + 1.38038065(4) c_{\text{SW}}^2 \right] \\
&+ \frac{g_o^4}{(16\pi^2)^2} c_F \left[ \ln^2(a_L^2 \bar{\mu}^2) \left( N_f + \frac{9}{2} c_F - \frac{11}{2} N_c \right) \right. \\
&\quad + \ln(a_L^2 \bar{\mu}^2) \left( -4.8388361(5) N_f + 2.3908354(3) N_f c_{\text{SW}} \right. \\
&\quad \quad - 14.13807433(4) N_f c_{\text{SW}}^2 \\
&\quad \quad + 81.0780218(9) c_F - 23.213749(2) c_F c_{\text{SW}} \\
&\quad \quad \left. + 4.1411425(3) c_F c_{\text{SW}}^2 + 37.4641674(9) N_c \right) \\
&\quad + N_f \left( 10.688(3) + 15.863(5) c_{\text{SW}} + 2.02(1) c_{\text{SW}}^2 \right. \\
&\quad \quad \left. + 22.078(3) c_{\text{SW}}^3 - 6.1807(8) c_{\text{SW}}^4 \right) \\
&\quad + c_F \left( -462.67(6) + 52.38(7) c_{\text{SW}} + 58.05(5) c_{\text{SW}}^2 \right. \\
&\quad \quad \left. - 27.759(4) c_{\text{SW}}^3 - 2.688(1) c_{\text{SW}}^4 \right) \\
&\quad + N_c \left( 36.93(4) - 132.40(5) c_{\text{SW}} - 4.04(3) c_{\text{SW}}^2 \right. \\
&\quad \quad \left. + 12.576(3) c_{\text{SW}}^3 + 1.0175(8) c_{\text{SW}}^4 \right) \Big] \tag{52}
\end{aligned}$$

$$\begin{aligned}
Z_P^{L,RI'} = 1 &+ \frac{g_o^2}{16\pi^2} c_F \left[ 3 \ln(a_L^2 \bar{\mu}^2) - 27.5954414(1) + 2.248868528(3) c_{\text{SW}} - 2.03601561(4) c_{\text{SW}}^2 \right] \\
&+ \frac{g_o^4}{(16\pi^2)^2} c_F \left[ \ln^2(a_L^2 \bar{\mu}^2) \left( \frac{9}{2} c_F + N_f - \frac{11}{2} N_c \right) \right. \\
&\quad + \ln(a_L^2 \bar{\mu}^2) \left( -8.1721694(4) N_f + 2.39083540(6) N_f c_{\text{SW}} \right. \\
&\quad \quad -14.13807433(4) N_f c_{\text{SW}}^2 \\
&\quad \quad +37.1489292(7) c_F + 6.746606(1) c_F c_{\text{SW}} \\
&\quad \quad \left. \left. -6.1080465(3) c_F c_{\text{SW}}^2 + 55.7975008(7) N_c \right) \right. \\
&\quad + N_f \left( 38.231(3) - 7.672(5) c_{\text{SW}} + 55.32(1) c_{\text{SW}}^2 \right. \\
&\quad \quad \left. -7.049(3) c_{\text{SW}}^3 + 4.7469(8) c_{\text{SW}}^4 \right) \\
&\quad + c_F \left( -876.98(4) + 84.52(2) c_{\text{SW}} + 38.65(4) c_{\text{SW}}^2 \right. \\
&\quad \quad \left. +19.974(3) c_{\text{SW}}^3 + 2.873(1) c_{\text{SW}}^4 \right) \\
&\quad + N_c \left( -104.35(3) - 38.06(2) c_{\text{SW}} - 14.57(3) c_{\text{SW}}^2 \right. \\
&\quad \quad \left. \left. -4.429(2) c_{\text{SW}}^3 - 1.2898(7) c_{\text{SW}}^4 \right) \right] \tag{53}
\end{aligned}$$

$$\begin{aligned}
Z_P^{L,\overline{MS}} = 1 &+ \frac{g_o^2}{16\pi^2} c_F \left[ 3 \ln(a_L^2 \bar{\mu}^2) - 14.5954414(1) + 2.248868528(3) c_{\text{SW}} - 2.03601561(4) c_{\text{SW}}^2 \right] \\
&+ \frac{g_o^4}{(16\pi^2)^2} c_F \left[ \ln^2(a_L^2 \bar{\mu}^2) \left( N_f + \frac{9}{2} c_F - \frac{11}{2} N_c \right) \right. \\
&\quad + \ln(a_L^2 \bar{\mu}^2) \left( 0.4944972(4) N_f + 2.39083540(6) N_f c_{\text{SW}} \right. \\
&\quad \quad -14.13807433(4) N_f c_{\text{SW}}^2 \\
&\quad \quad +76.1489292(7) c_F + 6.746606(1) c_F c_{\text{SW}} \\
&\quad \quad \left. \left. -6.1080465(3) c_F c_{\text{SW}}^2 + 8.1308341(7) N_c \right) \right. \\
&\quad + N_f \left( 16.013(3) + 2.688(5) c_{\text{SW}} - 5.94(1) c_{\text{SW}}^2 \right. \\
&\quad \quad \left. -7.049(3) c_{\text{SW}}^3 + 4.7469(8) c_{\text{SW}}^4 \right) \\
&\quad + c_F \left( -586.45(4) + 113.76(2) c_{\text{SW}} + 12.18(4) c_{\text{SW}}^2 \right. \\
&\quad \quad \left. +19.974(3) c_{\text{SW}}^3 + 2.873(1) c_{\text{SW}}^4 \right) \\
&\quad + N_c \left( 27.31(3) - 38.06(2) c_{\text{SW}} - 14.57(3) c_{\text{SW}}^2 \right. \\
&\quad \quad \left. \left. -4.429(2) c_{\text{SW}}^3 - 1.2898(7) c_{\text{SW}}^4 \right) \right] \tag{54}
\end{aligned}$$

All expressions reported thus far for  $Z_S$  and  $Z_P$  refer to flavor non singlet operators. In the case of  $Z_P$ , all diagrams of Fig.5 vanish, so that singlet and non singlet results coincide, just as in dimensional regularization. For  $Z_S$  on the other hand, the above diagrams give an additional finite contribution:

$$Z_{S,\text{singlet}}^{L,RI'} = Z_S^{L,RI'} + \frac{g_o^4}{(16\pi^2)^2} c_F N_f \left( -107.76(1) + 82.27(2) c_{\text{SW}} - 29.727(4) c_{\text{SW}}^2 \right. \\ \left. + 3.4400(7) c_{\text{SW}}^3 + 2.2758(4) c_{\text{SW}}^4 \right) \quad (55)$$

The same extra finite contribution applies also to the  $\overline{MS}$  scheme.

Finally, for completeness, and as an additional check on our results, we compute the renormalized Green's functions (for *vanishing* renormalized mass):

$$G_\psi^{RI'}(\bar{\mu}/q) \equiv Z_\psi^{L,RI'} \Sigma_{\text{odd}} \quad (56)$$

$$G_S^{RI'}(\bar{\mu}/q) \equiv Z_\psi^{L,RI'} Z_S^{L,RI'} \Sigma_S^L \quad (57)$$

$$G_P^{RI'}(\bar{\mu}/q) \equiv Z_\psi^{L,RI'} Z_P^{L,RI'} \Sigma_P^L \quad (58)$$

Similarly for  $\overline{MS}$ , taking into account Eq.(31).

Since these functions are regularization independent, they can be calculated also using, e.g., dimensional regularization. We have computed  $G_\psi$ ,  $G_S$  and  $G_P$  in both ways: either starting from our Eqs.(47-54) or using renormalization functions from dimensional regularization [1]. In all cases the two ways are in complete agreement. We obtain:

$$G_\psi^{RI'} = 1 + \frac{g_{RI'}^2}{16\pi^2} c_F \ln(\bar{\mu}^2/q^2) \\ + \frac{g_{RI'}^4}{(16\pi^2)^2} c_F \left[ \ln^2(\bar{\mu}^2/q^2) \left( \frac{1}{2} c_F + N_c \right) \right. \\ \left. + \ln(\bar{\mu}^2/q^2) \left( -\frac{19}{9} N_f - \frac{3}{2} c_F + \frac{251}{18} N_c \right) \right] \quad (59)$$

$$G_\psi^{\overline{MS}} = 1 + \frac{g_{\overline{MS}}^2}{16\pi^2} c_F \left[ \ln(\bar{\mu}^2/q^2) + 1 \right] \\ + \frac{g_{\overline{MS}}^4}{(16\pi^2)^2} c_F \left[ \ln^2(\bar{\mu}^2/q^2) \left( \frac{1}{2} c_F + N_c \right) \right. \\ + \ln(\bar{\mu}^2/q^2) \left( -N_f - \frac{1}{2} c_F + \frac{21}{2} N_c \right) \\ \left. + \left( -\frac{7}{4} N_f - \frac{5}{8} c_F + \left( \frac{143}{8} - 6\zeta(3) \right) N_c \right) \right] \quad (60)$$

$$G_S^{RI'} = 1 + \frac{g_{RI'}^2}{16\pi^2} c_F \left[ 4 \ln(\bar{\mu}^2/q^2) \right] + \frac{g_{RI'}^4}{(16\pi^2)^2} c_F \left[ \ln^2(\bar{\mu}^2/q^2) \left( -N_f + 8 c_F + \frac{13}{2} N_c \right) \right. \\ \left. + \ln(\bar{\mu}^2/q^2) \left( -\frac{58}{9} N_f + \frac{421}{9} N_c \right) \right] \quad (61)$$

Eq.(61) holds also for the case of the pseudoscalar operator:  $G_P^{RI'} = G_S^{RI'}$ .

$$\begin{aligned}
G_S^{\overline{MS}} = 1 &+ \frac{g_{\overline{MS}}^2}{16\pi^2} c_F \left[ 4 \ln(\bar{\mu}^2/q^2) + 6 \right] \\
&+ \frac{g_{\overline{MS}}^4}{(16\pi^2)^2} c_F \left[ \ln^2(\bar{\mu}^2/q^2) \left( -N_f + 8 c_F + \frac{13}{2} N_c \right) \right. \\
&\quad \left. + \ln(\bar{\mu}^2/q^2) \left( -\frac{16}{3} N_f + 24 c_F + \frac{130}{3} N_c \right) \right. \\
&\quad \left. + \left( -\frac{26}{3} N_f + \left( 22 + 12\zeta(3) \right) c_F + \left( \frac{227}{3} - 24\zeta(3) \right) N_c \right) \right] \quad (62)
\end{aligned}$$

$$\begin{aligned}
G_P^{\overline{MS}} = 1 &+ \frac{g_{\overline{MS}}^2}{16\pi^2} c_F \left[ 4 \ln(\bar{\mu}^2/q^2) + 14 \right] \\
&+ \frac{g_{\overline{MS}}^4}{(16\pi^2)^2} c_F \left[ \ln^2(\bar{\mu}^2/q^2) \left( -N_f + 8 c_F + \frac{13}{2} N_c \right) \right. \\
&\quad \left. + \ln(\bar{\mu}^2/q^2) \left( -\frac{16}{3} N_f + 56 c_F + \frac{130}{3} N_c \right) \right. \\
&\quad \left. + \left( -\frac{82}{9} N_f + \left( 134 + 12\zeta(3) \right) c_F + \left( \frac{679}{9} - 24\zeta(3) \right) N_c \right) \right] \quad (63)
\end{aligned}$$

In Figs. (6a,6b), (7a,7b), (8a,8b) we plot  $(Z_\psi^{L,\overline{MS}}, Z_\psi^{L,RI'})$ ,  $(Z_S^{L,\overline{MS}}, Z_S^{L,RI'})$  and  $(Z_P^{L,\overline{MS}}, Z_P^{L,RI'})$ , respectively, as a function of  $c_{\text{SW}}$ . In practice, of course, only specific values of  $c_{\text{SW}}$  are relevant, in the range  $1 \leq c_{\text{SW}} \leq 1.8$ , corresponding to perturbative or non-perturbative determinations. For definiteness, we have set  $N_c = 3$ ,  $\bar{\mu} = 1/a_L$  and  $\beta_o \equiv 2N_c/g_o^2 = 6.0$ . Our results up to two loops for each  $Z$  are shown for both  $N_f = 0$  and  $N_f = 2$ , and compared to the corresponding one-loop results. Furthermore, in the scalar case, we also present the two-loop result for the flavor singlet operator.

In Fig.9 we present, on the same plot, the values of  $Z_\psi^{L,\overline{MS}}$ ,  $Z_S^{L,\overline{MS}}$ ,  $Z_P^{L,\overline{MS}}$  and  $Z_{S,\text{singlet}}^{L,\overline{MS}}$  up to 2 loops, versus  $c_{\text{SW}}$ . We have chosen  $N_c = 3$ ,  $\bar{\mu} = 1/a_L$ ,  $N_f = 2$  and  $\beta_o = 5.3$ . The corresponding results in the  $RI'$  scheme are plotted in Fig.10.

There are a number of non-perturbative (NP) estimates of renormalization constants in the literature, in the  $RI'$  scheme (see, e.g., [38–40]) and in the Schrödinger functional scheme [41]. Our 2-loop results still differ from NP results in  $RI'$ , and this leaves open the possibility that higher loop effects may still be important, even though the perturbative series shows reasonable signs of convergence. A putative reason for this difference is the fact that the bare coupling constant  $g_o$  is known not to be a good expansion parameter. One may also express the renormalization functions in terms of the renormalized couplings:  $g_{\overline{MS}}$  or  $g_{RI'}$ . The resulting expressions for  $Z_S^{L,Y}$ ,  $Z_P^{L,Y}$  and  $Z_{S,\text{singlet}}^{L,Y}$  ( $Y = \overline{MS}, RI'$ ) as a function of  $c_{\text{SW}}$ , are shown in Figs.11 and 12, for the same values for  $N_c$ ,  $N_f$ ,  $\bar{\mu}$  and  $\beta_o$  as in Figs.9 and 10. For values of the clover parameter beyond its typical range,  $c_{\text{SW}} \geq 1.8$ , the behavior of the renormalization functions shows signs of instability at the scale  $\bar{\mu} = 1/a_L$ . There exist also several alternative definitions of an effective coupling in the literature; one should be aware, however, that the use of many of these definitions (coming, e.g., from boosted perturbation theory) can only be justified for 1-loop quantities, not beyond. For this reason, we have preferred to provide the bare results in this paper, leaving to the reader the straightforward task of converting these results to their favorite scheme.

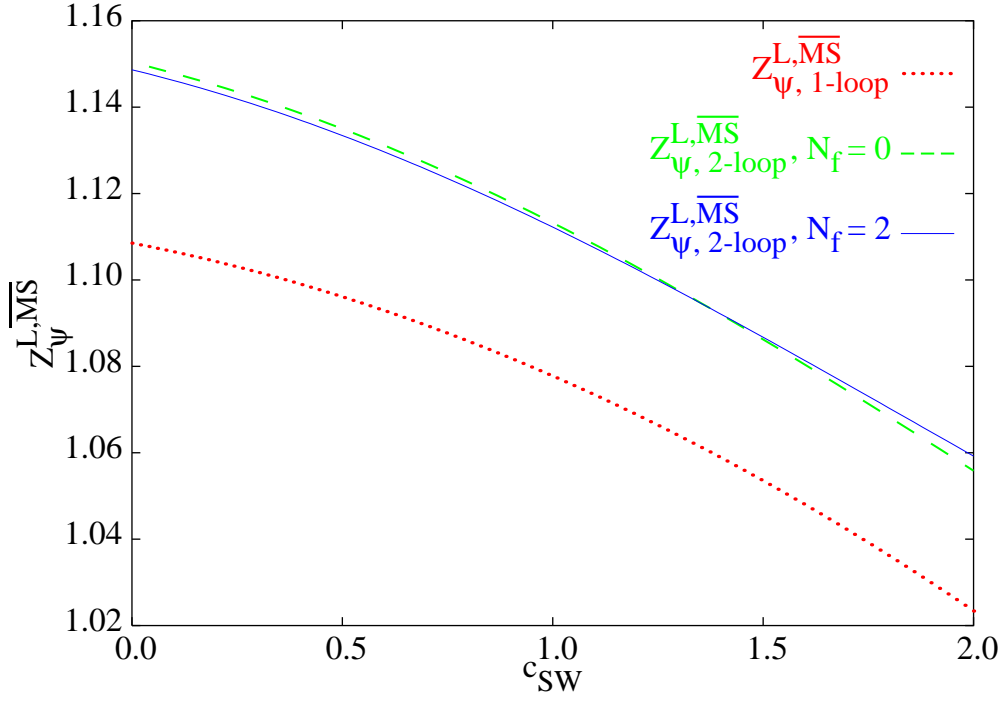


FIG. 6a.  $Z_{\psi}^{L,\overline{MS}}(a_L \bar{\mu})$  versus  $c_{\text{SW}}$  ( $N_c = 3$ ,  $\bar{\mu} = 1/a_L$ ,  $\beta_o = 6.0$ ). Results up to 2 loops are shown for  $N_f = 0$  (dashed line) and  $N_f = 2$  (solid line); one-loop results are plotted with a dotted line.

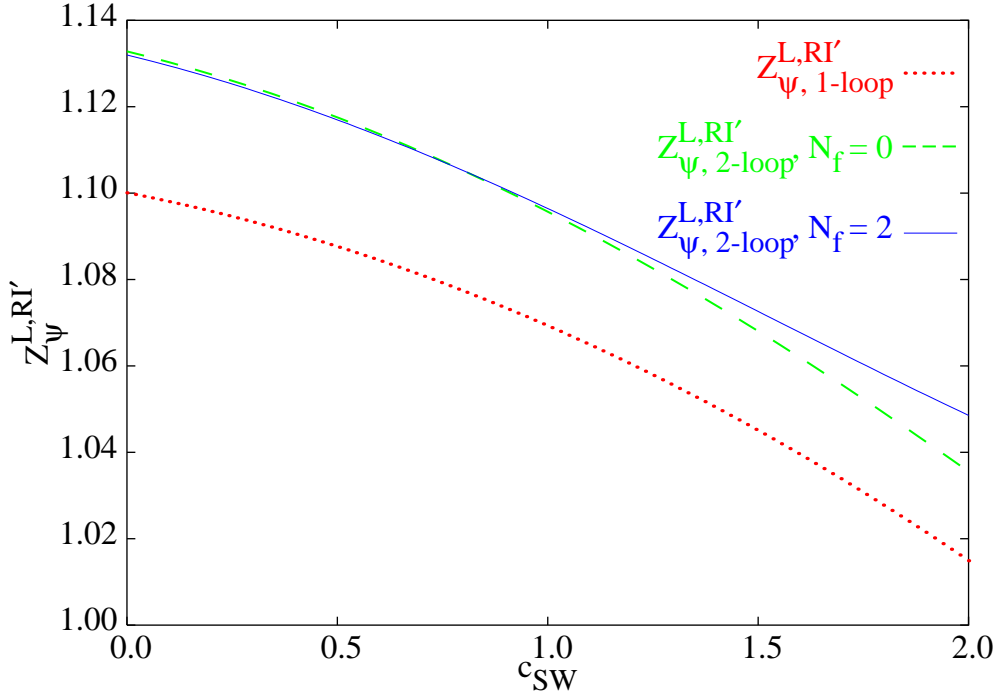


FIG. 6b.  $Z_{\psi}^{L,RI'}(a_L \bar{\mu})$  versus  $c_{\text{SW}}$  ( $N_c = 3$ ,  $\bar{\mu} = 1/a_L$ ,  $\beta_o = 6.0$ ). Same notation as in FIG.6a.

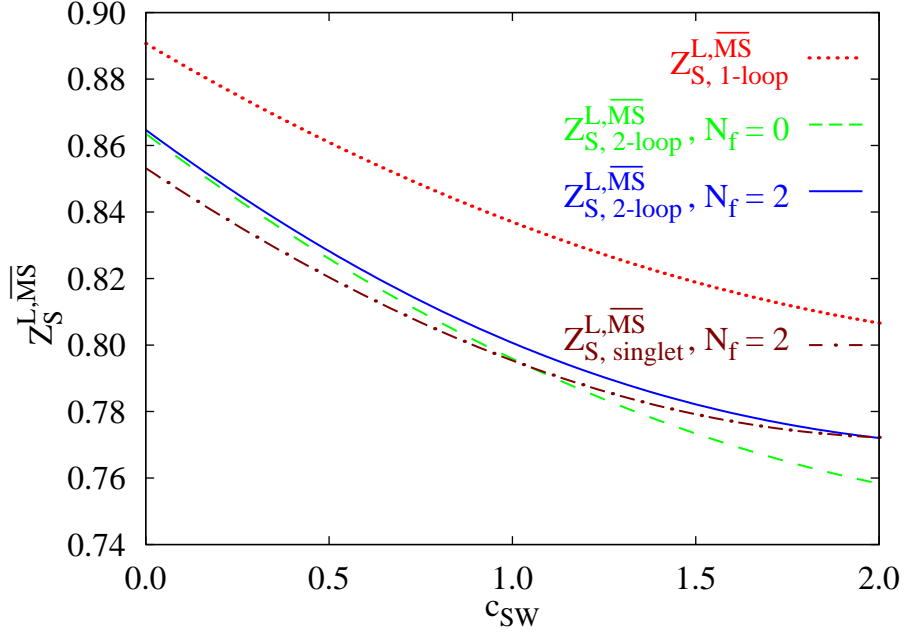


FIG. 7a.  $Z_S^{L, \overline{MS}}(a_L \bar{\mu})$  versus  $c_{SW}$  ( $N_c = 3$ ,  $\bar{\mu} = 1/a_L$ ,  $\beta_o = 6.0$ ). Results up to 2 loops, for the flavor non-singlet operator, are shown for  $N_f = 0$  (dashed line) and  $N_f = 2$  (solid line); 2-loop results for the flavor singlet operator, for  $N_f = 2$ , are plotted with a dash-dotted line; one-loop results are plotted with a dotted line.

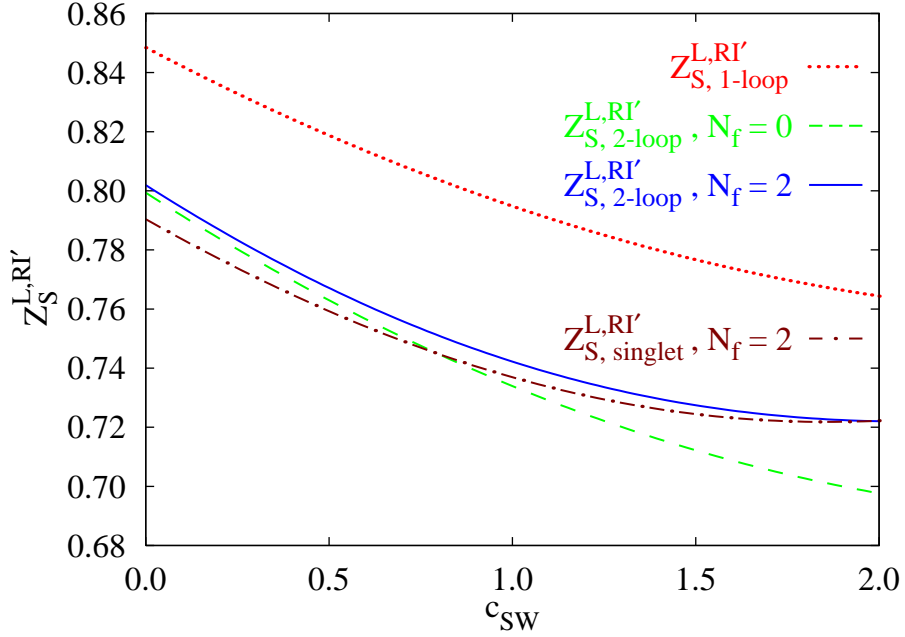


FIG. 7b.  $Z_S^{L, RI'}(a_L \bar{\mu})$  versus  $c_{SW}$  ( $N_c = 3$ ,  $\bar{\mu} = 1/a_L$ ,  $\beta_o = 6.0$ ). Same notation as in FIG.7a.

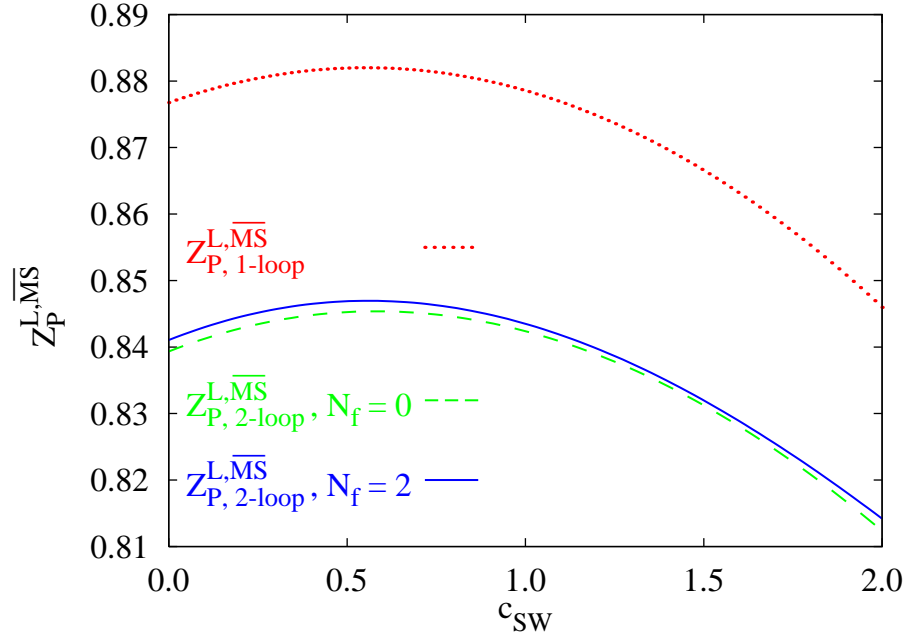


FIG. 8a.  $Z_P^{L, \overline{MS}}(a_L \bar{\mu})$  versus  $c_{SW}$  ( $N_c = 3$ ,  $\bar{\mu} = 1/a_L$ ,  $\beta_o = 6.0$ ). Results up to 2 loops are shown for  $N_f = 0$  (dashed line) and  $N_f = 2$  (solid line); one-loop results are plotted with a dotted line.

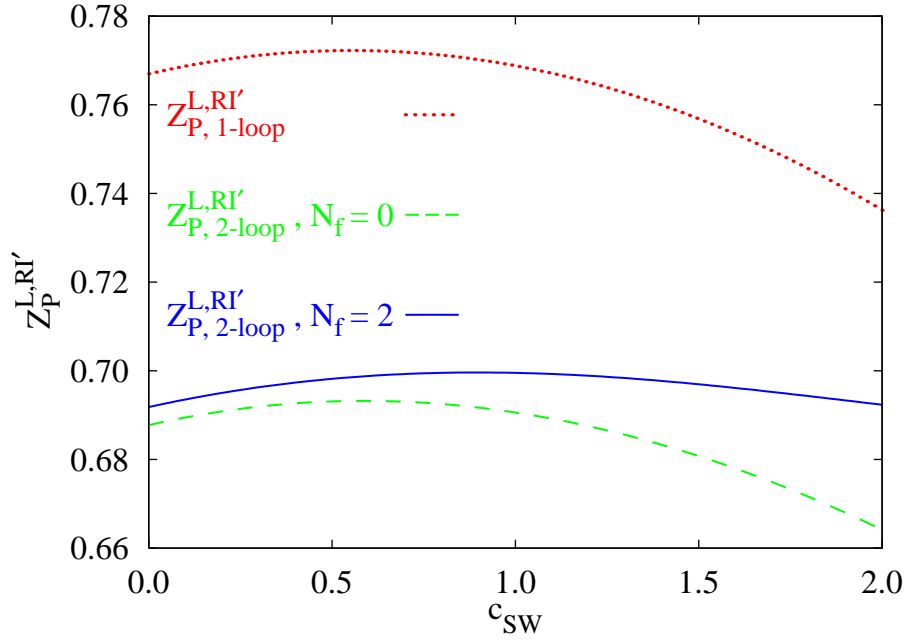


FIG. 8b.  $Z_P^{L, RI'}(a_L \bar{\mu})$  versus  $c_{SW}$  ( $N_c = 3$ ,  $\bar{\mu} = 1/a_L$ ,  $\beta_o = 6.0$ ). Same notation as in FIG.8a.

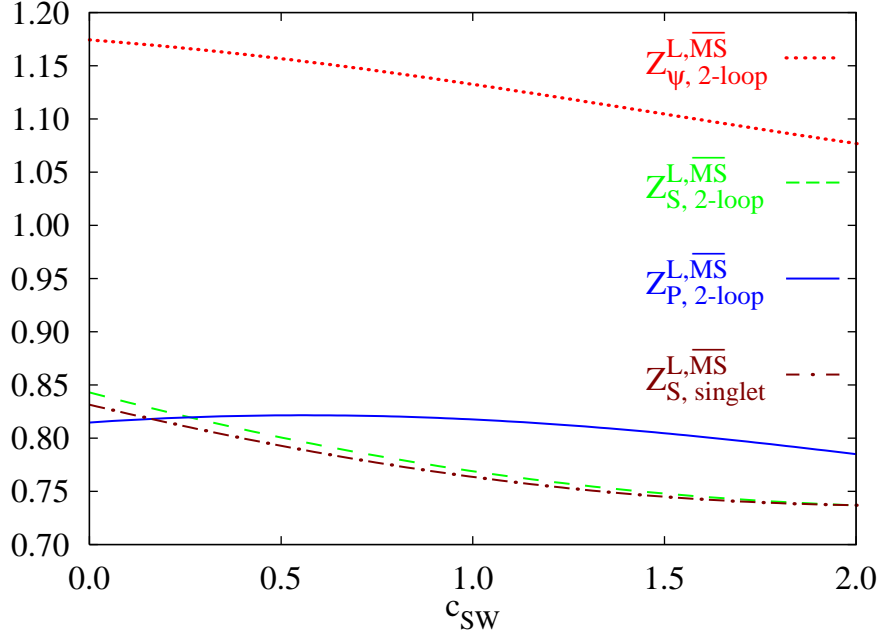


FIG. 9.  $Z_{\psi}^{L, \overline{MS}}$  (dotted line),  $Z_S^{L, \overline{MS}}$  (dashed line),  $Z_P^{L, \overline{MS}}$  (solid line) and  $Z_{S, singlet}^{L, \overline{MS}}$  (dash-dotted line) up to 2 loops, versus  $c_{SW}$  ( $N_c = 3$ ,  $\bar{\mu} = 1/a_L$ ,  $N_f = 2$ ,  $\beta_o = 5.3$ ).

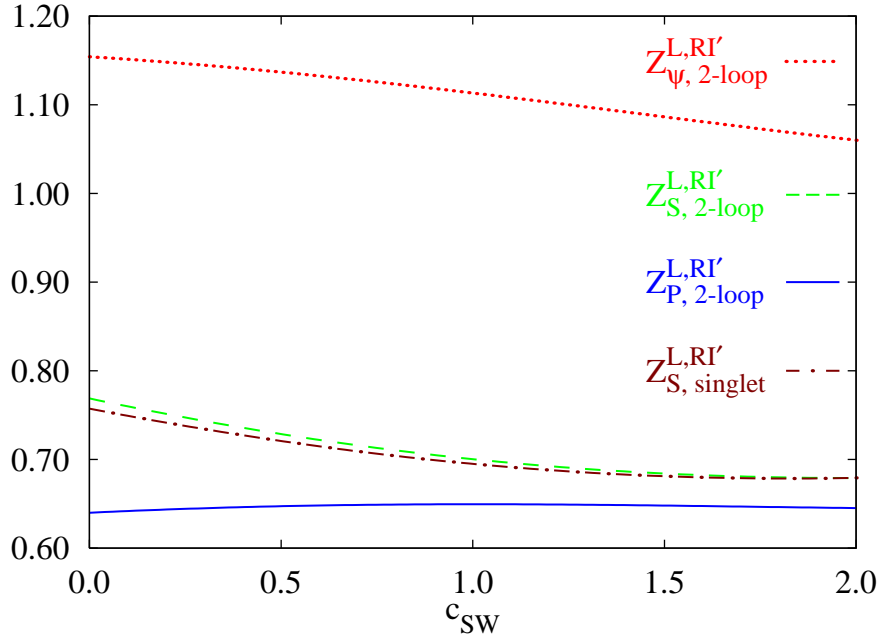


FIG. 10.  $Z_{\psi}^{L, RI'}$  (dotted line),  $Z_S^{L, RI'}$  (dashed line),  $Z_P^{L, RI'}$  (solid line) and  $Z_{S, singlet}^{L, RI'}$  (dash-dotted line) up to 2 loops, versus  $c_{SW}$  ( $N_c = 3$ ,  $\bar{\mu} = 1/a_L$ ,  $N_f = 2$ ,  $\beta_o = 5.3$ ).



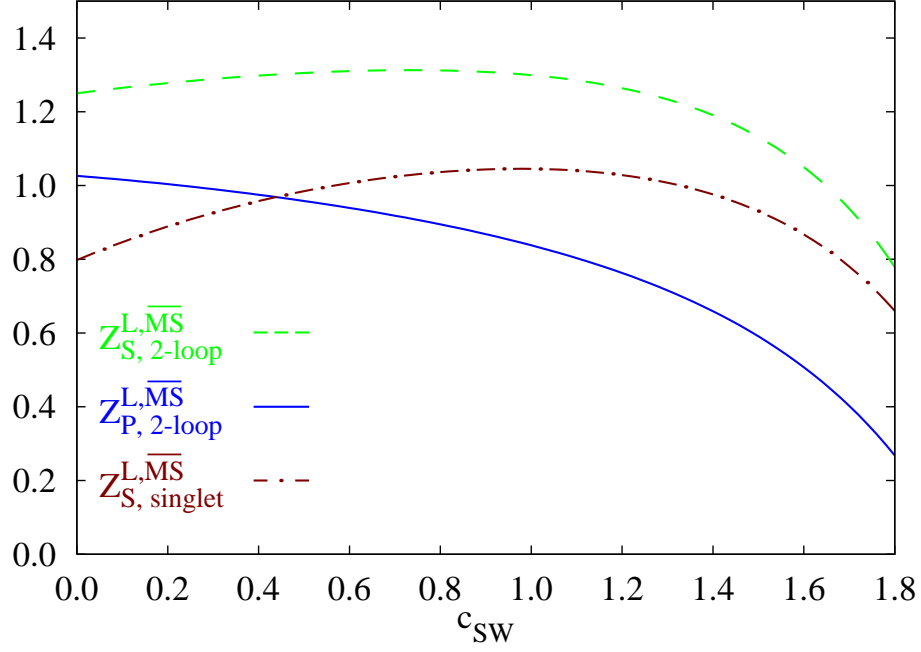


FIG. 11. 2-loop renormalization functions  $Z_S^{L,\overline{MS}}$  (dashed line),  $Z_P^{L,\overline{MS}}$  (solid line) and  $Z_{S,singlet}^{L,\overline{MS}}$  (dash-dotted line) expressed in terms of the renormalized coupling constant  $g_{\overline{MS}}$ , versus  $c_{SW}$  ( $N_c = 3$ ,  $\bar{\mu} = 1/a_L$ ,  $N_f = 2$ ,  $\beta_o = 5.3$ ).

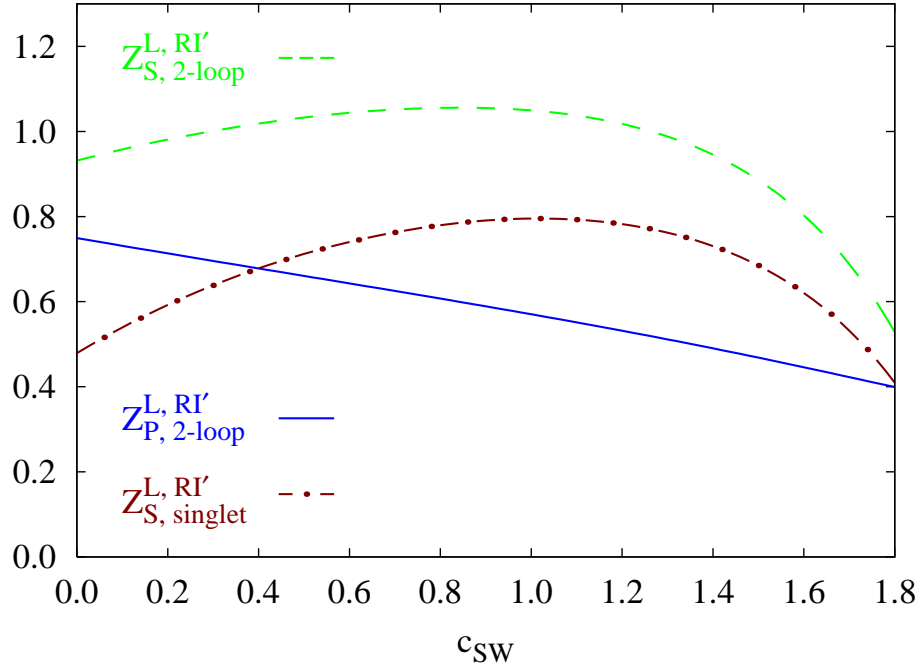


FIG. 12. 2-loop renormalization functions  $Z_S^{L,RI'}$  (dashed line),  $Z_P^{L,RI'}$  (solid line) and  $Z_{S,singlet}^{L,RI'}$  (dash-dotted line) expressed in terms of the renormalized coupling constant  $g_{RI'}$ , versus  $c_{SW}$  ( $N_c = 3$ ,  $\bar{\mu} = 1/a_L$ ,  $N_f = 2$ ,  $\beta_o = 5.3$ ).

## IV. DISCUSSION

As can be seen from Figs.6a-8b, all 2-loop renormalization functions differ from 1-loop values in a significant way; this difference should be taken into account in MC simulations, in order to reduce systematic error. At the same time, 2-loop contributions are consistently smaller than 1-loop contributions, indicating that the (asymptotic) perturbative series are under control.

The dependence on the clover parameter  $c_{\text{SW}}$  is also quite pronounced. In the present work,  $c_{\text{SW}}$  was left as a free parameter; its optimal value, as dictated by  $\mathcal{O}(a_L)$  improvement, has been estimated both non-perturbatively [42] and perturbatively (to 1-loop) [19].

Our results regard both the flavor nonsinglet and singlet operators. For the pseudoscalar operator, these cases coincide, just as in dimensional regularization. The scalar operator, on the other hand, receives an additional finite ( $a_L \bar{\mu}$  independent) contribution in the flavor singlet case.  $Z_{S, \text{singlet}}$  is seen to be equal to the fermion mass renormalization  $Z_m$ , which is an essential ingredient in the computation of quark masses.

We note also that, in dimensional regularization, both the scalar and pseudoscalar flavor singlet operators renormalize in the same way as their non-singlet counterparts, for mass independent renormalization schemes. Consequently, the conversion factors  $C_S$  and  $C_P$ , as well as  $Z_5$ , stay the same for flavor singlets.

A breakdown of our results on a *per diagram* basis has not been presented here, due to lack of space; it is available from the authors upon request.

The 2-loop computation of the renormalization functions for the Vector, Axial and Tensor bilinears is work currently in progress.

Besides the strictly local definitions of fermion bilinears,  $\bar{\psi}\Gamma\psi$ , one can consider a family of more extended operators (see, e.g., [42]), with the same classical continuum limit, as dictated by  $\mathcal{O}(a_L)$  improvement. The renormalization of these extended operators involves more Feynman diagrams, since their vertices may also contain gluon lines; however, the computation is actually less cumbersome, since all additional contributions are now free of superficial divergences. We will be reporting the results of this computation in a future work.

## APPENDIX: Fermions in an arbitrary representation

Our results for  $Z_\psi$ ,  $Z_S$ ,  $Z_P$ , Eqs.(49, 51, 53), can be easily generalized to an action with Wilson/clover fermions in an arbitrary representation  $R$ , of dimensionality  $d_R$ .

In this case, the gluon part of the action remains the same, while all link variables appearing in the fermion part of the action assume the form:

$$U_{x, x+\mu} = \exp(i g_0 A_\mu^a(x) T^a) \longrightarrow U_{x, x+\mu} = \exp(i g_0 A_\mu^a(x) T_R^a) \quad (64)$$

Using standard notation and conventions, the generators  $T^a$  in the fundamental representation satisfy:

$$[T^a, T^b] = i f^{abc} T^c, \quad \sum_a T^a T^a \equiv \mathbb{1}_{c_F} = \mathbb{1} \frac{N_c^2 - 1}{2N_c}, \quad \text{tr}(T^a T^b) \equiv \delta^{ab} t_F = \delta^{ab} \frac{1}{2} \quad (65)$$

In the representation  $R$  we have:

$$[T_R^a, T_R^b] = i f^{abc} T_R^c, \quad \sum_a T_R^a T_R^a \equiv \mathbb{1}_{c_R}, \quad \text{tr}(T_R^a T_R^b) \equiv \delta^{ab} t_R \quad (66)$$

where:  $t_R = (d_R c_R)/(N_c^2 - 1)$ .

For the 1-loop quantities, Eqs.(47, 48), converting to the representation  $R$  is a straightforward substitution:

$$N_f \longrightarrow N_f \cdot (2 t_R) \quad (67)$$

and, in addition, for Eqs.(41-46):

$$c_F \longrightarrow c_R \quad (68)$$

Aside from these changes, all algebraic expressions (and the numerical coefficients resulting from loop integrations) remain the same.

A similar reasoning applies to the 2-loop quantities in Eqs.(49, 51, 53): For most diagrams, once their value is expressed as a linear combination of  $c_F^2$ ,  $c_F N_c$  and  $c_F N_f$ , it suffices to apply substitutions (67) and (68). The only exceptions are diagrams containing a gluon tadpole [diagrams 7, 14 (Fig.2); diagram 3 (Fig.4); 1-loop diagrams, when expressed in terms of  $a_{RI'}$ ,  $\alpha_{RI'}$  by means of  $Z_g$ ,  $Z_A$ ]: In these cases, only one power of  $c_F$  should be changed to  $c_R$ ; a possible additional power of  $c_F$  originates from a gluon tadpole and should stay as is. This peculiarity implies that, in order to perform the substitutions as described above, one must start from the *per diagram* breakdown of 2-loop results. To avoid presenting a lengthy breakdown, we apply, instead, substitutions (67) and (68) indiscriminately on Eqs.(49, 51, 53); consequently, we must then add a correction term, as follows:

$$\begin{aligned}
Z_{\psi}^{L,RI'}|_R &= Z_{\psi}^{L,RI'}|_{c_F \rightarrow c_R, N_f \rightarrow 2N_f t_R} \\
&+ \frac{g_o^4}{(16\pi^2)^2} c_R (c_R - c_F) \cdot [-4\pi^2 \ln(a_L^2 \bar{\mu}^2) - 467.9141661(2) \\
&+ 88.7817709(1) c_{\text{SW}} + 55.1618942(2) c_{\text{SW}}^2] \quad (69)
\end{aligned}$$

$$\begin{aligned}
Z_S^{L,RI'}|_R &= Z_S^{L,RI'}|_{c_F \rightarrow c_R, N_f \rightarrow 2N_f t_R} \\
&+ \frac{g_o^4}{(16\pi^2)^2} c_R (c_R - c_F) \cdot [-12\pi^2 \ln(a_L^2 \bar{\mu}^2) + 708.732752(6) \\
&+ 305.48068(1) c_{\text{SW}} - 54.495244(2) c_{\text{SW}}^2] \quad (70)
\end{aligned}$$

$$\begin{aligned}
Z_P^{L,RI'}|_R &= Z_P^{L,RI'}|_{c_F \rightarrow c_R, N_f \rightarrow 2N_f t_R} \\
&+ \frac{g_o^4}{(16\pi^2)^2} c_R (c_R - c_F) \cdot [-12\pi^2 \ln(a_L^2 \bar{\mu}^2) + 1089.424358(4) \\
&- 88.7817709(1) c_{\text{SW}} + 80.378675(2) c_{\text{SW}}^2] \quad (71)
\end{aligned}$$

[Actually, the reader could arrive at these results without knowledge of the *per diagram* breakdown, by virtue of the following fact: All ‘exceptional’ powers of  $c_F$  cancel out of  $Z_{\psi}^{L,RI'}$ ,  $Z_S^{L,RI'}$ ,  $Z_P^{L,RI'}$ , if these are expressed in terms of the renormalized coupling constant  $a_{RI'}$ . Thus, one may:

- Express Eqs.(49, 51, 53) in terms of  $g_{RI'}$  by means of  $g_o = (Z_g^{L,RI'}) g_{RI'}$ , with  $Z_g^{L,RI'}$  in the fundamental representation (Eq.(48))
- Apply substitutions (67), (68) throughout
- If desired, reexpress everything in terms of  $g_o$  (using  $(Z_g^{L,RI'})^{-1}$  from Eq.(48), with  $N_f \rightarrow 2N_f t_R$  and  $c_F$  as is)

No correction terms are necessary in this procedure.]

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